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**Moving Average Models for
Interest Rates and Applications to
Life Insurance Mathematics**

by

D Velmachos and S Haberman

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**Department of Actuarial Science and Statistics
City University
London**

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Abstract

The paper examines the effect on standard functions from life insurance mathematics when not only mortality but also interest rates are treated as having a random component. The analysis is concerned with a particular type of stochastic model, viz the force of interest is modelled by an unconditional moving average process of order q . These explicit results may be seen as extensions to those of Frees (1990) who considered the case $q=1$.

1. INTRODUCTORY COMMENTS

Historically, the theory of life contingencies has developed from deterministic beginnings.

Random fluctuations in mortality, morbidity, interest and expenses, were ignored, although actuaries have implicitly attempted to allow for random fluctuations by using conservative assumptions for each of the factors entering a formulae. For example, in the calculation of the present values of the liabilities for a policy, we can assume that mortality follows an a priori known mortality table or that the variability due to mortality (i.e. variability in future lifetimes) can be ignored because of the presence of a very large number of identical liabilities in respect of different lives. Similarly, the interest rate may be assumed to be constant or an implicit allowance may be made by adopting a conservative estimate of future interest rates.

A first step forward in the development of the subject was to consider the time until decrement (death, disability, and so on) as a random variable in the calculation of actuarial functions, while the interest rate was assumed to be constant. This is a "semi-stochastic approach".

It is only since about 1970 that there has been interest in actuarial models which consider both the time until decrement and the investment rate of return as random variables.

Pollard (1971) and Boyle (1976) have considered interest rate fluctuations by treating the force of interest as a random variable. Boyle (1976) examined the case in which the force of interest in any year is a normal variable but uncorrelated with the force of interest in any other year. This is a natural and simple assumption to make and is explored further in section 2.

Pollard (1971) on the other hand modelled the force of interest by using a particular stationary autoregressive process of order two. Panjer and Bellhouse (1980) and Bellhouse and Panjer (1981) have developed a general theory for autoregressive models of the force of interest, including both continuous and discrete models. This theory has been developed for unconditional and conditional autoregressive processes of order one and two.

Giaccotto (1986) has developed an algorithm for evaluating present value functions when interest rates are assumed to follow an ARIMA process. Also Wilkie (1976), Waters (1978), Westcott (1981), de Jong (1984), Dhaene (1989) and Frees (1990) have considered stochastic interest models in the calculation of the standard actuarial functions of life insurance mathematics.

The application of such time series based stochastic interest rate models has become the response that actuaries provide to the criticism of other financial analysts that the "traditional" actuarial approach does not take into account the uncertainty of future values of interest rates.

A question that now arises is whether the stochastic nature which is used for the calculation of interest rates is correct. Many actuaries remain sceptical as to the use of a stochastic interest rate model believing that the results obtained owe more to the specific model used than to any underlying reality. This question of the significance of model sensitivity is the subject of

current research and interested readers are referred to Wright (1997) for an investigation in the area of pension funding models.

However, in this paper, we will concentrate on models with a certain stochastic nature and the choice of an incorrect interest model will not be considered here.

As in Frees (1990), we use the sequence $\{\Delta_k\}$ to model a stochastic environment. Here, $\{\Delta_k\}$ represents the random force of interest in the k -th period. $\{\Delta_k\}$ can be interpreted as a one-period spot rate.

It is convenient to model the force of interest as a random quantity, in lieu of the effective interest or discount rate, due to the linear nature of correlation and autoregressive models and the multiplicative nature of compound interest.

In this paper, we consider only discrete time models, and for convenience we refer to time intervals as years.

2. INDEPENDENT INTEREST RATE MODEL

2.1 Introduction

In this section we will consider the case in which the $\{\Delta_k\}$ are independent, identically distributed (i.i.d) random variables.

With the interpretation of the $\{\Delta_k\}$ as one-period spot rates and the i.i.d assumption, then at time 0 the present value of one unit payable at time k is:

$$v_k = \prod_{s=1}^k \exp(-\Delta_s) = \exp\left(-\sum_{s=1}^k \Delta_s\right)$$

so that the logarithm of v_k is a random walk. This is a feature desirable from the viewpoint of the theory of financial economics because the random walk is a special case of a martingale, the structure of which does not permit riskless arbitrage (see Frees (1990)): this is a requirement of this theory. The i.i.d assumption also provides a benchmark assumption against which more complex models, such as those of section 3, can be compared.

We will also assume that $\{\Delta_k\}$ are distributed normally with mean μ and variance σ^2 so that $\Delta_k \sim N(\mu, \sigma^2)$. In this case $\exp(-\Delta_k)$ is said to be lognormally distributed with parameters $-\mu, \sigma^2$. Then, $\exp(-\Delta_k) \sim \log N(-\mu, \sigma^2)$.

2.2 Some Basic Results

The moment generating function of Δ_k is

$$E(\exp(t \Delta_k)) = M_\Delta(t) = \exp(t\mu + t^2 \sigma^2 / 2).$$

Thus with $t = -1$:

$$E(\exp(-\Delta_k)) = M_\Delta(-1) = \exp(-\mu + \sigma^2 / 2). \quad (2.1)$$

The present value at time 0 of 1 unit payable at time k is :

$$v_k = e^{-\sum_{i=1}^k \Delta_i} \text{ for } k \geq 1 \text{ and } v_k = 1 \text{ for } k = 0.$$

So the expected value of the present value is:

$$\begin{aligned} E(v_k) &= E(\exp(-\sum_{i=1}^k \Delta_i)) = E(\exp(-\Delta_1) \exp(-\Delta_2) \dots \exp(-\Delta_k)) \stackrel{\Delta \text{ i.i.d}}{=} \\ &= E(\exp(-\Delta_1)) E(\exp(-\Delta_2)) \dots E(\exp(-\Delta_k)) = (\exp(\mu - \sigma^2 / 2))^k = \exp(-k(\mu - \sigma^2 / 2)). \end{aligned}$$

$$\text{We set } d_1 = \mu - \sigma^2 / 2 \text{ then } E(v_k) = \exp(-k d_1). \quad (2.2)$$

$$v_k^2 = \exp(-2 \sum_{i=1}^k \Delta_i) \Rightarrow E(v_k^2) = E(\exp(-2 \sum_{i=1}^k \Delta_i)) = (M_\Delta(-2))^k = \exp(-k 2(\mu - \sigma^2 / 2)).$$

We set $d_2 = 2(\mu - \sigma^2 / 2)$ and then

$$E(v_k^2) = \exp(-k d_2). \quad (2.3)$$

$$\text{In general : } E(v_k^n) = E(\exp(-n \sum_{i=1}^k \Delta_i)) = (M_\Delta(-n))^k = \exp(-k(n\mu - n^2 \sigma^2 / 2))$$

$$\text{and so } E(v_k^n) = \exp(-k d_n) \text{ where } d_n = n\mu - n^2 \sigma^2 / 2. \quad (2.4)$$

$$\text{Similarly, we find that: } E(v_s v_r) = \exp(-r d_1) \exp(-s(d_2 - d_1)) \quad (s < r). \quad (2.5)$$

2.3 Life Insurance Actuarial Functions

This section is based on the approach of Frees (1990). We will assume that there is only one decrement, mortality. We define the integer-valued random variable K to be the curtate time of decrement, so that if T_x is the future time until decrement of a person aged x then $K < T_x < K + 1$.

We will use the standard notation: $P(K = k) = {}_k q_x$ and $P(K > k) = {}_{k+1} p_x$ $k = 0, 1, 2, \dots$ for the discrete probability function and survival function respectively.

Also we will assume that K is independent of $\{\Delta_k\}$.

Two types of general contracts are considered: insurance and annuity contracts.

For the general insurance contract, a benefit b_{k+1} is taken to be payable at time $k+1$ at the end of the year of death, given that death occurs during the year $(k, k+1)$.

The random present value of this insurance benefit is $Z_{K+1} = v_{K+1} b_{K+1}$ where $b_{K+1} \in (-\infty, +\infty)$.

For the general annuity contract, payments c_s are payable at time s , at the beginning of each year up to and including the year of death.

The random present value of the annuity benefits is then:

$$a(K) = \sum_{s=0}^K v_s c_s \text{ where } c_s \in (-\infty, +\infty).$$

By the law of conditional expectations, we have

$E[g(X)] = E[E[g(X)/Y]]$ so for the case of the life insurance contract:

$$\begin{aligned} E(Z_{K+1}) &= E[E(v_{K+1} b_{K+1} / K=k)] = E[\exp(-(K+1)d_1) b_{K+1}] = \\ &= \sum_{k=0}^{\infty} e^{-d_1(k+1)} b_{k+1} q_x. \end{aligned} \quad (2.6)$$

$$\begin{aligned} E(Z_{K+1}^2) &= E[E(v_K^2 b_{K+1}^2 / K=k)] = E[\exp(-(K+1)d_2) b_{K+1}^2] = \\ &= \sum_{k=0}^{\infty} e^{-d_2(k+1)} b_{k+1}^2 q_x. \end{aligned} \quad (2.7)$$

In a similar manner, we can use (2.4) to demonstrate that

$$E(Z_{K+1}^n) = \sum_{k=0}^{\infty} e^{-d_n(k+1)} b_{k+1}^n q_x \quad (2.8)$$

For the case of the annuity contract:

$$E[a(K)] = E[E(\sum_{s=0}^K v_s c_s / K=k)] = E[\sum_{s=0}^K e^{-d_1 s} c_s] = \sum_{s=0}^{\infty} c_s e^{-d_1 s} p_x. \quad (2.9)$$

$$E[a(K)^2] = E(\sum_{s=0}^K e^{-d_2 s} c_s^2) + 2 E(\sum_{r=1}^K \sum_{s=0}^{r-1} e^{-d_1 s} e^{-(d_2-d_1)r} c_r c_s). \quad (2.10)$$

$$E[Z_{K+1} a(K)] = E(b_{K+1} \sum_{s=0}^K e^{-s(d_2-d_1)} e^{-(K+1)d_1} c_s). \quad (2.11)$$

As an alternative approach, Waters (1978) provides simple expressions for the moments of compound interest functions which are useful for calculating the high order moments of certain actuarial functions which will be used later in some numerical examples.

Let $\tilde{a}_{n|}$ be the stochastic equivalent for the annuity-certain $a_{n|}$ then $\tilde{a}_{n|} = \sum_{k=1}^n v_k$.

Let $\theta = \exp(-\mu)$ $\phi = \exp(\sigma^2/2)$, then

$$E(\tilde{a}_{n|}) = \sum_{k=1}^n \theta^k \phi^k \quad (2.12)$$

and the following results are obtained by Waters (1978):

$$E(\tilde{a}_{n|}^2) = \sum_{k=1}^n \theta^{2k} \phi^{4k} + 2 \sum_{k=2}^n \sum_{j=1}^{k-1} \theta^{k+j} \phi^{(k+j)} \quad (2.13)$$

$$E(\tilde{a}_{n|}^3) = \sum_{k=1}^n \theta^{3k} \phi^{9k} + 3 \sum_{k=2}^n \sum_{j=1}^{k-1} (\theta^{2k+j} \phi^{(4k+5j)} + \theta^{k+2j} \phi^{(k+8j)}) + 6 \sum_{k=3}^n \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} \theta^{i+j+k} \phi^{(5i+3j+k)} \quad (2.14)$$

$$E(\tilde{a}_{n|}^4) = \sum_{k=1}^n \theta^{4k} \phi^{16k} + 4 \sum_{k=2}^n \sum_{j=1}^{k-1} (\theta^{k+3j} \phi^{(k+15j)} + \theta^{3k+j} \phi^{(9k+7j)}) + 6 \sum_{k=2}^n \sum_{j=1}^{k-1} \theta^{2k+2j} \phi^{(4k+12j)} + 12 \sum_{k=3}^n \sum_{j=2}^{k-1} \sum_{i=1}^{j-1} (\theta^{i+j+2k} \phi^{(7i+5j+4k)} + \theta^{i+2j+k} \phi^{(7i+8j+k)} + \theta^{2i+j+k} \phi^{(12i+3j+k)}) + 24 \sum_{k=4}^n \sum_{j=3}^{k-1} \sum_{i=2}^{j-1} \sum_{h=1}^{i-1} \theta^{h+i+j+k} \phi^{(7h+5i+3j+k)} \quad (2.15)$$

$$E(\tilde{a}_{n|} v_m) = \sum_{k=1}^n \theta^{m+k} \phi^{(m+3k)} \quad \text{if } m > n$$

$$= \sum_{k=1}^m \theta^{m+k} \phi^{(m+3k)} + \sum_{k=1}^n \theta^{m+k} \phi^{(k+3m)} \quad \text{if } m < n.$$

We note also that

\tilde{A}_x (Whole Life Assurance), $\tilde{A}_{x:n|}^1$ (Temporary Assurance)
 $\tilde{A}_{x:n|}$ (Endowment Assurance), \tilde{a}_x (Immediate Whole Life Annuity)
 \tilde{a}_x (Whole Life Annuity due), $\tilde{a}_{x:n|}$ (Immediate Temporary Annuity)

are the stochastic equivalents (i.e random variables) corresponding to the standard actuarial functions. In order to derive expressions for the moments of these random variables we use the rules of conditional expectation, as applied earlier.

We let $g(K)$ be the present value of the benefit for a standard life insurance policy issued to a life aged x . Then, for the case of a whole life policy,

$$g(K) = \{v_{K+1} \mid K \geq 0\} \quad \text{and} \quad E[g(K)^m] = \sum_{k=0}^{\infty} k/q_x E[v_{k+1}^m]. \quad (2.16)$$

For the case of a temporary insurance policy:

$$g(K) = \begin{cases} v_{K+1} & K < n \\ 0 & K \geq n \end{cases} \quad \text{and} \quad E[g(K)^m] = \sum_{k=0}^{n-1} k/q_x E[v_{k+1}^m]. \quad (2.17)$$

For the case of an endowment insurance policy: $g(K) = \begin{cases} v_{K+1} & K < n \\ v_n & K \geq n \end{cases}$ and

$$E[g(K)^m] = \sum_{k=0}^{n-1} k/q_x E[v_{k+1}^m] + {}_np_x E[v_n^m]. \quad (2.18)$$

We let $h(K)$ be the present value of a standard annuity policy issued to a life aged x . Then, for the case of an immediate whole life annuity:

$$E[h(K)^m] = \sum_{k=1}^{\infty} k/q_x E[\tilde{a}_k^m]. \quad (2.19)$$

For the case of an immediate temporary annuity then:

$$E[h(K)^m] = \sum_{k=1}^{n-1} k/q_x E[\tilde{a}_k^m] + {}_np_x E[\tilde{a}_n^m].$$

Also we know that the present value of the whole life annuity-due is equal to the present value of the immediate whole life annuity plus 1 and the present value of the temporary annuity-due (age x , term n) is equal to the present value of the immediate temporary annuity (age x , term $n-1$) plus 1.

2.4 Examples

The above formulae (2.6-2.20) enable us to compute the moments of the present values and related functions for these simple insurances and annuities.

We will assume in the following calculations in this section and in section 3.6 that all lives are subject to the mortality experience of A 1967-70 (Ultimate).

Tables 2.1 - 2.5, on the following pages, present calculated values of the expectation, standard deviation, skewness and kurtosis for the case of $\mu=0.07$ and for several values of x , n , (where appropriate) and σ for the respective cases of an endowment, temporary and whole life insurance, and a temporary and whole life annuity.

Some relevant graphs are also presented (Figures 2.1 - 2.3).

When $\sigma = 0$ the stochastic equivalents of the actuarial functions are equal to the normal actuarial functions calculated on a deterministic basis..

In each example, the standard deviation of the actuarial function increases as σ increases (as we can see in Fig. 2.1 for the Temporary Insurance), but the amount of increase is not significant. This feature occurs because the greater part of the variation in these functions is due to fluctuations in the age at death as opposed to fluctuations in the interest rates. Of course, this situation changes if we consider a large number of independent lives and then the fluctuations in interest rates would become more important.

The standard measures of skewness and kurtosis are calculated as $\frac{\mu_3}{\mu_2^{3/2}}$ and $\frac{\mu_4}{\mu_2^2}$ respectively

where μ_n is the n^{th} central moment of the distribution. So, for a generic random variable X , we have that:

$$\text{Skewness of } X = \dots = \frac{E(X^3) - 3E(X^2)E(X) + 2E(X)^3}{VAR(X)^{3/2}}$$

$$\text{Kurtosis of } X = \dots = \frac{E(X^4) - 4E(X^3)E(X) + 6E(X^2)E(X)^2 - 3E(X)^4}{VAR(X)^2}.$$

We can see from the Tables and Figures that the values of skewness and kurtosis are both very large. This reflects the very skew and very sharply peaked shape of the density functions.

Figures 2.2 and 2.3 also show that an increase in σ is associated with decreases in the absolute values of the skewness and kurtosis. This is because, when σ increases, the density function will tend to be spread more evenly over its range.

Endowment Assurance

$\mu = 0.07$

Age
▼

20	30	40	20	30	40
----	----	----	----	----	----

← Term

$\sigma = 0$

$\sigma = 0.03$

25	0.250732	0.130961	0.075913	0.252958	0.132643	0.077098
30	0.252214	0.135393	0.08456	0.254441	0.137091	0.085801
35	0.255969	0.144352	0.099852	0.258195	0.146076	0.101177
40	0.263217	0.159945	0.124037	0.26544	0.161706	0.125479
45	0.275708	0.184779	0.159124	0.277924	0.186594	0.160713

Expectation

25	0.040116	0.054966	0.065193	0.052351	0.059008	0.066894
30	0.043825	0.062549	0.075287	0.055305	0.066334	0.077113
35	0.054743	0.07873	0.093565	0.064397	0.082066	0.095499
40	0.0721	0.101686	0.117287	0.079803	0.104664	0.119368
45	0.094606	0.128931	0.143257	0.100758	0.131727	0.145534

Standard Deviation

25	12.16509	9.478377	7.736863	5.611329	7.815982	7.334613
30	10.2824	7.407445	5.818319	5.255787	6.36953	5.580754
35	7.768355	5.463463	4.315624	4.887615	4.958812	4.19439
40	5.698423	4.026704	3.264449	4.286018	3.795818	3.199596
45	4.171295	2.977109	2.499181	3.511257	2.867781	2.4601

Skewness

25	165.9879	106.9121	76.5245	59.53698	82.9287	71.18357
30	122.6754	68.83172	46.31082	50.67303	56.48902	43.71835
35	71.7015	38.79347	26.66794	39.35196	34.27791	25.64169
40	39.41627	21.9352	16.07935	27.59368	20.41533	15.65257
45	21.8678	12.83286	10.20655	17.85284	12.30768	10.00679

Kurtosis

$\sigma = 0.05$

$\sigma = 0.07$

25	0.256965	0.13569	0.079256	0.263096	0.140397	0.082616
30	0.258449	0.140165	0.088056	0.264581	0.144912	0.091564
35	0.262201	0.149194	0.103582	0.26833	0.154005	0.107315
40	0.26944	0.164891	0.128094	0.275559	0.1698	0.13214
45	0.281912	0.189871	0.163589	0.288009	0.194915	0.168025

Expectation

25	0.070038	0.066259	0.070151	0.092462	0.077413	0.075626
30	0.072355	0.073203	0.08058	0.094324	0.083908	0.086342
35	0.079653	0.088218	0.099146	0.100186	0.098	0.105149
40	0.092743	0.110202	0.123267	0.111106	0.119116	0.129624
45	0.111535	0.136932	0.149776	0.127524	0.145327	0.156635

Standard Deviation

25	2.698064	5.816389	6.668243	1.744333	4.141832	5.800229
30	2.68798	5.0323	5.187335	1.765969	3.815507	4.674605
35	2.864348	4.238725	3.992629	1.921236	3.488757	3.727957
40	2.935804	3.441217	3.092483	2.08443	3.031094	2.954378
45	2.728892	2.694033	2.396992	2.097574	2.483059	2.319456

Skewness

25	21.15711	55.42012	62.40491	10.07848	33.57412	51.07552
30	19.86285	41.04694	39.44889	9.942408	27.41702	33.91595
35	19.09772	27.8804	23.93996	10.40519	21.2425	21.7187
40	16.92069	18.06497	14.9508	10.49592	15.3121	14.05671
45	13.16966	11.46342	9.686215	9.432219	10.42355	9.302338

Kurtosis

(Table 2.1)

Temporary Insurance

$\mu = 0.07$

Age

20	30	40	20	30	40
----	----	----	----	----	----

← Term

$\sigma = 0$

$\sigma = 0.03$

25	0.008802	0.015809	0.025669	0.00884	0.015927	0.025941
30	0.013122	0.025255	0.040547	0.013187	0.025457	0.040988
35	0.022484	0.042475	0.064463	0.022599	0.042814	0.065144
40	0.039462	0.070528	0.099116	0.039662	0.071074	0.100106
45	0.067541	0.112467	0.144796	0.067875	0.113299	0.146125

Expectation

25	0.06884	0.076295	0.079364	0.069311	0.077065	0.08039
30	0.080006	0.090104	0.092185	0.080649	0.091164	0.093552
35	0.102151	0.112944	0.111433	0.103023	0.114334	0.113138
40	0.133218	0.141417	0.133424	0.134365	0.14316	0.13544
45	0.170499	0.171063	0.154938	0.171965	0.173165	0.157226

Standard Deviation

25	8.953197	6.755749	5.724027	8.979497	6.745778	5.678192
30	6.969304	4.922426	4.16433	6.998017	4.922136	4.134205
35	5.097343	3.55962	3.145786	5.12375	3.563889	3.124647
40	3.683315	2.603871	2.498685	3.704725	2.607934	2.479716
45	2.630119	1.902682	2.058647	2.647343	1.905402	2.039504

Skewness

25	91.00436	58.89308	47.46284	91.45446	58.5813	46.68274
30	56.76734	33.43919	27.64519	57.13931	33.28769	27.179
35	31.20699	18.67758	17.00241	31.46779	18.62261	16.73788
40	17.07158	11.08223	11.54311	17.22666	11.05037	11.36838
45	9.522994	7.047666	8.39353	9.611581	7.020165	8.268706

Kurtosis

$\sigma = 0.05$

$\sigma = 0.07$

25	0.00891	0.016139	0.026435	0.009016	0.016464	0.027198
30	0.013303	0.025819	0.041786	0.013481	0.026374	0.043018
35	0.022806	0.043424	0.066378	0.023119	0.044358	0.068282
40	0.040021	0.072057	0.101896	0.040567	0.073562	0.104653
45	0.068476	0.114796	0.148527	0.069388	0.117088	0.152222

Expectation

25	0.070165	0.078477	0.082292	0.092462	0.077413	0.075626
30	0.081814	0.093106	0.096083	0.094324	0.083908	0.086342
35	0.104604	0.116881	0.116295	0.100186	0.098	0.105149
40	0.136446	0.146354	0.139173	0.111106	0.119116	0.129624
45	0.174624	0.177017	0.161462	0.127524	0.145327	0.156635

Standard Deviation

25	9.031695	6.734988	5.604097	9.123981	6.737299	5.513491
30	7.054487	4.928545	4.088527	7.152976	4.956173	4.040998
35	5.175067	3.577167	3.093909	5.2631	3.611769	3.065846
40	3.746162	2.619827	2.45199	3.816844	2.649694	2.426167
45	2.680645	1.91433	2.011029	2.737365	1.938216	1.982863

Skewness

25	92.4058	58.14589	45.39441	94.24417	57.83147	43.74943
30	57.92817	33.11777	26.43958	59.45547	33.13837	25.57846
35	32.01151	18.58814	16.32832	33.04223	18.71085	15.8816
40	17.54876	11.03288	11.09886	18.15655	11.11486	10.80867
45	9.796604	6.996641	8.076328	10.14826	7.032748	7.870315

Kurtosis

(Table 2.2)

Whole Life Assurance

$\mu = 0.07$

Age
▼

$\sigma = 0$

$\sigma = 0.03$

25	0.046933
30	0.062972
35	0.085592
40	0.116086
45	0.155641

0.047692
0.063904
0.086719
0.117421
0.157185

Expectation

25	0.0739
30	0.083739
35	0.100591
40	0.122126
45	0.145857

0.075033
0.08518
0.102317
0.124116
0.148103

Standard Deviation

25	6.241921
30	4.808983
35	3.769096
40	3.008606
45	2.398801

6.131449
4.725328
3.710624
2.966109
2.366687

Skewness

25	55.43961
30	34.97892
35	22.08254
40	14.50455
45	9.765005

53.88551
33.98406
21.53847
14.20148
9.591821

Kurtosis

$\sigma = 0.05$

$\sigma = 0.07$

25	0.04908
30	0.065604
35	0.088768
40	0.119843
45	0.159978

0.051254
0.068256
0.091954
0.123594
0.164288

Expectation

25	0.077178
30	0.087896
35	0.105554
40	0.127834
45	0.152282

0.080743
0.092365
0.110841
0.133868
0.159025

Standard Deviation

25	5.938401
30	4.583711
35	3.613735
40	2.896926
45	2.315474

5.662286
4.392437
3.48828
2.810612
2.25439

Skewness

25	51.18419
30	32.30988
35	20.6411
40	13.70986
45	9.316718

47.35429
30.07196
19.49152
13.10493
8.996194

Kurtosis

(Table 2.3)

Temporary Annuity

$\mu = 0.07$

Age

20	30	40	20	30	40
----	----	----	----	----	----

Term

$\sigma = 0$

$\sigma = 0.03$

25	10.3242	11.96872	12.71783	10.36248	12.02552	12.78646	Expectation
30	10.29898	11.89758	12.58322	10.33708	11.95367	12.65011	
35	10.23709	11.75601	12.34796	10.2748	11.81078	12.41202	
40	10.11901	11.51195	11.97955	10.15599	11.56452	12.03943	
45	9.917046	11.12658	11.45024	9.952814	11.17582	11.5045	

25	0.618771	0.83109	0.975919	0.941087	1.232491	1.409108	Standard Deviation
30	0.679557	0.948195	1.127272	0.981233	1.311098	1.510452	
35	0.850512	1.192883	1.398156	1.105092	1.492901	1.711886	
40	1.118759	1.536578	1.747301	1.320614	1.773098	1.994143	
45	1.463675	1.940734	2.127609	1.62034	2.124356	2.318852	

25	-11.8378	-9.26744	-7.60488	-3.20914	-2.66065	-2.32707	Skewness
30	-9.91057	-7.20182	-5.70944	-3.13693	-2.54443	-2.17538	
35	-7.45813	-5.30802	-4.24368	-3.26187	-2.54737	-2.13481	
40	-5.46917	-3.91813	-3.22227	-3.21176	-2.41616	-2.01927	
45	-4.0046	-2.90335	-2.47788	-2.86105	-2.10398	-1.79248	

25	157.9609	102.9104	74.46677	31.19068	22.93274	18.77073	Kurtosis
30	114.7629	65.62365	44.96435	27.91833	19.39841	15.35806	
35	66.62529	36.97874	26.01169	24.54627	16.16924	12.64791	
40	36.66236	21.01373	15.80142	19.68412	12.61449	10.08173	
45	20.41433	12.385	10.1075	14.11653	9.149648	7.707116	

$\sigma = 0.05$

$\sigma = 0.07$

25	10.43104	12.12753	12.90996	10.53511	12.28306	13.09889	Expectation
30	10.40533	12.05439	12.77046	10.50892	12.20794	12.9545	
35	10.34233	11.90911	12.52724	10.44484	12.05901	12.70336	
40	10.22223	11.65891	12.14709	10.32276	11.80275	12.31153	
45	10.01687	11.2642	11.60199	10.11408	11.39883	11.75076	

25	1.346338	1.750425	1.98179	1.812798	2.359162	2.666662	Standard Deviation
30	1.3735	1.802147	2.04441	1.831577	2.392051	2.699517	
35	1.462496	1.931385	2.181837	1.896588	2.481522	2.78386	
40	1.628275	2.145003	2.388515	2.022877	2.636896	2.9188	
45	1.874877	2.429016	2.638916	2.219585	2.851979	3.088018	

25	-0.80788	-0.59959	-0.48214	0.064039	0.210994	0.295443	Skewness
30	-0.86139	-0.65989	-0.53392	0.026046	0.164144	0.249355	
35	-1.15124	-0.88719	-0.71997	-0.16288	-0.00165	0.102133	
40	-1.49783	-1.11829	-0.90806	-0.45788	-0.23156	-0.09767	
45	-1.6726	-1.20324	-0.99373	-0.73506	-0.42551	-0.2761	

25	9.591583	7.830894	7.015083	5.320939	4.942853	4.820204	Kurtosis
30	9.30434	7.462155	6.603544	5.272849	4.862703	4.721178	
35	9.72376	7.461756	6.484426	5.545584	4.937625	4.725592	
40	9.825548	7.180295	6.212775	5.881287	4.9792	4.704384	
45	8.800274	6.29753	5.582212	5.841431	4.767332	4.520536	

(Table 2.4)

Whole of Life Annuity

$\mu = 0.07$

Age
▼

$\sigma = 0$

$\sigma = 0.03$

25	13.09734
30	12.8601
35	12.52551
40	12.07446
45	11.48938

25	13.17409
30	12.93279
35	12.59322
40	12.13625
45	11.54441

Expectation

25	1.093088
30	1.238624
35	1.487895
40	1.806433
45	2.157448

25	1.535606
30	1.625412
35	1.803931
40	2.055049
45	2.349702

Standard Deviation

25	-6.24192
30	-4.80899
35	-3.7691
40	-3.00861
45	-2.39881

25	-2.03259
30	-1.91441
35	-1.9269
40	-1.88862
45	-1.73235

Skewness

25	55.43965
30	34.97896
35	22.08257
40	14.50457
45	9.765023

25	15.87259
30	13.20502
35	11.30506
40	9.441296
45	7.493314

Kurtosis

$\sigma = 0.05$

$\sigma = 0.07$

25	13.31247
30	13.06377
35	12.71511
40	12.2474
45	11.64332

25	13.52481
30	13.26451
35	12.90171
40	12.41732
45	11.79431

Expectation

25	2.134861
30	2.17524
35	2.282771
40	2.454176
45	2.671955

25	2.863422
30	2.85975
35	2.90236
40	2.993636
45	3.125019

Standard Deviation

25	-0.38029
30	-0.43745
35	-0.6262
40	-0.83292
45	-0.951

25	0.36631
30	0.313379
35	0.162477
40	-0.0467
45	-0.24375

Skewness

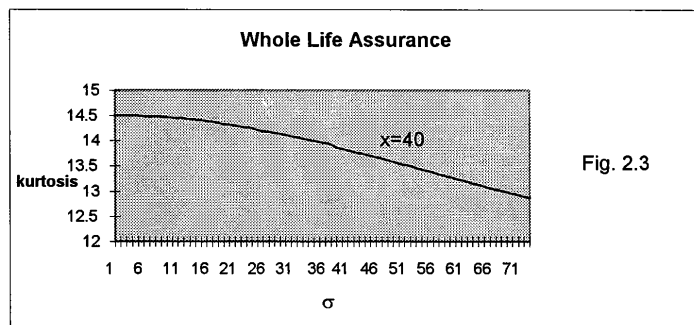
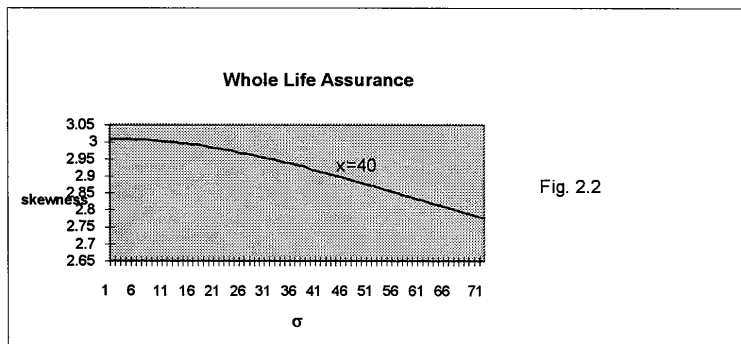
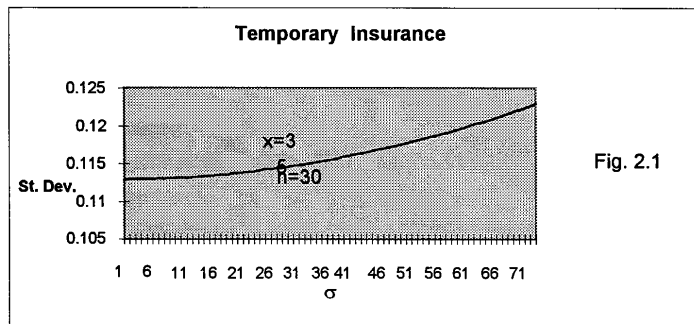
25	6.496247
30	6.166195
35	6.124692
40	5.981172
45	5.485222

25	4.793086
30	4.696396
35	4.689028
40	4.669524
45	4.505019

Kurtosis

(Table 2.5)

Charts



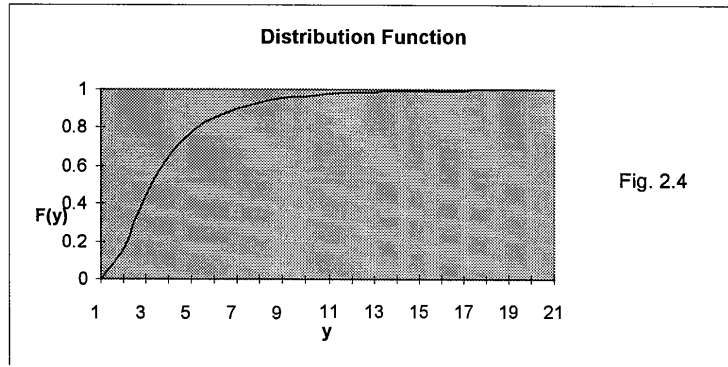
2.6 Cumulative Distribution Function

The approach can be used to derive the cumulative distribution function (and hence the probability density function) for the random variables defined in section 2.3 as in Frees (1990). As an illustration of the method, we derive an expression for and calculate the cumulative distribution function for the present value of a whole life assurance, \tilde{A}_x . Then, following Frees (1990):

$$F(y) = P(\tilde{A}_x \leq y) = E [P(v_{K+1} \leq y / K=k)] = \sum_{k=1}^{\infty} P(v_{k+1} \leq y) {}_kq_x =$$

$$= \sum_{k=1}^{\infty} \Phi\left(\frac{\log y + (k+1)\mu}{\sigma\sqrt{k+1}}\right) {}_kq_x$$

The following graph gives the Cum. Distr. Fun. of \tilde{A}_x for $x=30$, $\mu=0.05$, $\sigma=0.07$



The median is 0.119019, but the mean is 0.138238.

The fact that the median is less than the mean is an indication that the distribution is skewed to the right (as noted earlier).

2.6 Comments

In this section, formulae for the mean, variance, skewness, kurtosis and high order moments have been developed, for several types of life insurance and annuity contract, in the presence of independent and identically distributed investment returns.

However, we have to note that the i.i.d. normality assumption will rarely be satisfied in practice. It does though serve as a useful benchmark. In the next section, we introduce a moving average representation of the force of interest.

3. DEPENDENT INTEREST RATE MODEL: The MOVING AVERAGE PROCESS MA(q)

3.1 Introduction

As Frees (1990) mentions, "... the assumption that the interest environment represented by the sequence $\{\Delta_k\}$ is i.i.d. is a useful modification of the traditional assumption that $\{\Delta_k\}$ is deterministic. This modification permits volatility of interest rates in the model".

In this section, we continue with this approach and assume that the interest rate model is stochastic but also dependent. It is assumed that the force of interest follows a moving average model of order q (q is a variable). In this section we will generalise some of the ideas of Frees (1990) and discuss the general case of the $MA(q)$ model rather than the specific case of $q=1$.

This model $\{MA(q)\}$ accounts for certain autocorrelation aspects of the sequence $\{\Delta_k\}$ and has the advantage of being tractable (in the mathematical sense) in terms of the calculation of insurance functions. The model has also been applied to pension funding problems by Haberman and Wong (1997) and Bédard and Dufresne (1998).

3.2 Basic Results

We now consider the unconditional $MA(q)$ model,

$$\Delta_k = \mu + \varepsilon_k + a_1 \varepsilon_{k-1} + a_2 \varepsilon_{k-2} + \dots + a_q \varepsilon_{k-q} . \quad (3.1)$$

where $\{\varepsilon_k\}_{k=-\infty}^{+\infty}$ is an i.i.d. sequence with mean zero and variance σ^2 . The coefficients $\{a_i\}(i=1,2,\dots,q)$ are usually constrained so that the roots of the characteristic equation $1 - \sum_{i=1}^q a_i x^i = 0$ lie outside the unit circle. If $q=1$, this constraint becomes $-1 < a_1 < 1$, while if $q=2$, the conditions become:

$$a_1 + a_2 < 1, \quad a_2 - a_1 < 1, \quad -1 < a_2 < 1.$$

These constraints are required so that the model is invertible, that is it can be alternatively expressed as an autoregressive model of infinite order, or as the limit of a sequence of autoregressive models of finite order - the invertibility requirement ensures that this sequence is convergent (Box and Jenkins, 1976). Invertibility is similar to the stationarity condition

applied to autoregressive models. We define $M(t) = E[e^{t'\varepsilon}]$ to be the moment generating function of ε and assume that $M(t)$ exists.

The following proposition is an important building block for this section.

3.2.1 Proposition 1

$$E[v_k] = C_l e^{-kg_l} \quad \text{for } k=1,2,\dots$$

$$g_l = \mu - \ln(M(-(1 + \sum_{i=1}^q a_i))) \quad (3.2)$$

$$C_l = \frac{\left(\prod_{j=1}^{q-1} M(-(1 + \sum_{i=1}^j a_i)) \right) M(-1) \left(\prod_{j=1}^q M(-\sum_{i=j}^q a_i) \right)}{\left(M(-(1 + \sum_{i=1}^q a_i)) \right)^q}. \quad (3.3)$$

Proof

$$\Delta_1 = \mu + \varepsilon_1 + a_1 \varepsilon_0 + \dots + a_q \varepsilon_{1-q}$$

$$\Delta_2 = \mu + \varepsilon_2 + a_1 \varepsilon_1 + \dots + a_q \varepsilon_{2-q}$$

.

.

$$\Delta_k = \mu + \varepsilon_k + a_1 \varepsilon_{k-1} + \dots + a_q \varepsilon_{k-q}$$

By adding the above (and collecting terms along diagonals) we have:

$$\begin{aligned} \sum_{i=1}^k \Delta_i &= k\mu + \sum_{i=1}^{k-q} \varepsilon_i (1 + a_1 + \dots + a_q) + (1 + a_1 + \dots + a_{q-1}) \varepsilon_{k-q+1} + \\ &+ (1 + a_1 + \dots + a_{q-2}) \varepsilon_{k-q+2} + \dots + (1 + a_1) \varepsilon_{k-1} + \varepsilon_k + (a_1 + a_2 + \dots + a_q) \varepsilon_0 + \\ &+ (a_2 + \dots + a_q) \varepsilon_1 + \dots + (a_{q-1} + a_q) \varepsilon_{2-q} + a_q \varepsilon_{1-q}. \\ v_k &= e^{-\sum_{i=1}^k \Delta_i} = e^{-k\mu} e^{-(1+a_1+\dots+a_q)\sum_{i=1}^{k-q} \varepsilon_i} e^{-(1+a_1+\dots+a_{q-1})\varepsilon_{k-q+1}} \dots e^{-(1+a_1)\varepsilon_{k-1}} e^{-\varepsilon_k} \dots \\ &\dots e^{-(a_1+\dots+a_q)\varepsilon_0} \dots e^{-(a_{q-1}+a_q)\varepsilon_{2-q}} e^{-a_q \varepsilon_{1-q}}. \end{aligned} \quad (3.4)$$

Each component of v_k is independent so that we can write:

$$\begin{aligned}
E[v_k] &= e^{-k\mu} E[e^{-\sum_{i=1}^{k-q} (1+a_1+\dots+a_q)\varepsilon_i}] E[e^{-(1+a_1+\dots+a_{q-1})\varepsilon_{k-q+1}}] \dots E[e^{-(1+a_1)\varepsilon_{k-1}}] E[e^{-\varepsilon_k}] \dots \\
&\dots E[e^{-(a_1+\dots+a_q)\varepsilon_0}] \dots E[e^{-(a_{q-1}+a_q)\varepsilon_{2-q}}] E[e^{-a_q\varepsilon_{1-q}}] = \\
&= e^{-k\mu} (M(-(1+\dots+a_q)))^{k-q} M(-(1+\dots+a_{q-1})) \dots \\
&\dots M(-1)M(-(a_1+\dots+a_q)) \dots M(-a_q) = \\
&= e^{-k\mu} e^{k\ln(M(-(1+\dots+a_q)))} \frac{M(-(1+\dots+a_{q-1})) \dots M(-1)M(-(a_1+\dots+a_q)) \dots M(-a_q)}{(M(-(1+\dots+a_q)))^q} \quad (3.5)
\end{aligned}$$

$= e^{-k(\mu - \ln(M(-(1+\dots+a_q))))} C_1 = C_1 e^{-kg_1}$ on introducing the above definitions of g_1 and C_1 .

Note that in the special case of $a_1 = a_2 = \dots a_q = 0$, then $g_1 = d_1$ leading to the i.i.d result given earlier in section 2.

3.2.2 Proposition 2

$$i) E[v_k^n] = C_n e^{-kg_n} \quad n \geq 1 \quad (3.6)$$

$$ii) E[v_s v_k] = B e^{-sg_2} e^{-(k-s)g_1} \quad s < k \quad (3.7)$$

where $g_n = n\mu - \ln(M(-n(1 + \sum_{i=1}^q a_i)))$

$$C_n = \frac{\left(\prod_{j=1}^{q-1} M(-nS_j) \right) M(-n) \left(\prod_{j=1}^q M(-n(S_q - S_{j-1})) \right)}{(M(-nS_q))^q} \quad (3.8)$$

$$B = \frac{\left(\prod_{l=0}^{q-1} M(-2(S_l + \frac{1}{2}(S_q - S_l))) \right) M(-1) \left(\prod_{j=1}^q M(-2(S_q - S_{j-1})) \right) \left(\prod_{j=1}^{q-1} M(-S_j) \right)}{(M(-2S_q))^q (M(-S_q))^q} \quad (3.9)$$

and we use the additional shorthand notation $S_0 = 1$ and $S_j = 1 + \sum_{i=1}^j a_i$ for $j \geq 1$.

Proof

i) From equation (3.4), we obtain directly:

$$E[v_k^n] = e^{-k[\mu - \ln(M(-n(1+a_1+\dots+a_{q-1})))]} \frac{M(-n(1+a_1+\dots+a_{q-1})) \cdots M(-n)M(-n(a_1+\dots+a_q)) \cdots M(-na_q)}{M(-n(1+a_1+\dots+a_q))^q}$$

which reduces to (3.6) an substitution for g_n and S_j .

$$\text{ii) If } s < k: E[v_s v_k] = E[e^{-\sum_{i=1}^s \Delta_i} e^{-\sum_{i=s+1}^k \Delta_i}] = E[e^{-\left(\sum_{i=1}^s \Delta_i + \sum_{i=s+1}^k \Delta_i\right)}]$$

Then, proceeding in the same manner as for the proof of Proposition 1, we write:

$$\begin{aligned} \sum_{i=1}^s \Delta_i + \sum_{i=s+1}^k \Delta_i &= 2 \sum_{i=1}^s \Delta_i + \sum_{i=s+1}^k \Delta_i = (k-s)\mu + 2s\mu + 2(1+a_1+\dots+a_q) \sum_{i=1}^{s-q} \varepsilon_i + \\ &+ (2(1+\dots+a_{q-1})+a_q) \varepsilon_{s-q+1} + (2(1+\dots+a_{q-2})+a_{q-1}+a_q) \varepsilon_{s-q+2} + \dots + \\ &+ (2+a_1+\dots+a_q) \varepsilon_s + (1+\dots+a_q) \sum_{i=s+1}^{k-q} \varepsilon_i + (1+\dots+a_{q-1}) \varepsilon_{k-q+1} + \dots + \varepsilon_k + \\ &+ 2(a_1+\dots+a_q) \varepsilon_0 + 2(a_2+\dots+a_q) \varepsilon_1 + \dots + 2a_q \varepsilon_{1-q}. \end{aligned}$$

Thus we can write

$$\begin{aligned} E[v_s v_k] &= B e^{-s(2\mu - \ln(M(-2(1+a_1+\dots+a_q))))} e^{-(k-s)(\mu - \ln(M(-(1+\dots+a_q))))} \\ &= B e^{-sg_1} e^{-(k-s)g_1} \end{aligned}$$

on substituting for g_1 and g_2 and with B given by

$$\begin{aligned} B &= M(-2(1+\dots+a_{q-1})-a_q)M(-2(1+\dots+a_{q-2})-a_{q-1}-a_q) \cdots M(-(2+a_1+\dots+a_q)) \times \\ &\frac{M(-(1+\dots+a_{q-1})) \cdots M(-1)M(-2(a_1+\dots+a_q))M(-2(a_2+\dots+a_q)) \cdots M(-2a_q)}{M(-2(1+a_1+\dots+a_q))^q M(-(1+a_1+\dots+a_q))^q}. \end{aligned}$$

As before if $a_1 = a_2 = \dots a_q = 0$, then $g_n = d_n$ and $C_n = 1$ as in the i.i.d result of section 2.

3.3 Life Insurance Actuarial Functions

If we consider, as in section 2, the general insurance and annuity contracts with random present values, $Z_{K+1} = v_{K+1} b_{K+1}$ and $a(K) = \sum_{s=0}^K v_s c_s$ respectively, we can calculate again the mean and variance of Z_{K+1} and $a(K)$.
Thus,

Thus,

$$E[Z_{K+1}] = C_1 E[e^{-g_1(K+1)} b_{K+1}] \quad (3.10)$$

$$\begin{aligned} E[a(K)] &= E\left[\sum_{s=0}^K v_s c_s\right] = E\left[c_0 + \sum_{s=1}^K v_s c_s\right] = c_0 + E\left[\sum_{s=1}^K v_s c_s\right] = \\ &= c_0 + C_1 E\left[\sum_{s=1}^K e^{-g_1 s} c_s\right] \end{aligned} \quad (3.11)$$

and

$$E[Z_{K+1}^2] = C_2 E[e^{-g_2(K+1)} b_{K+1}^2] \quad (3.12)$$

$$\begin{aligned} E[a(K)^2] &= E\left[\left(c_0 + \sum_{s=1}^K v_s c_s\right)^2\right] = E\left[c_0^2 + 2c_0 \sum_{s=1}^K v_s c_s + \left(\sum_{s=1}^K v_s c_s\right)^2\right] = \\ &= c_0^2 + 2c_0 E\left[\sum_{s=1}^K v_s c_s\right] + E\left[\left(\sum_{s=1}^K v_s c_s\right)^2\right] \end{aligned} \quad (3.13)$$

where $E\left[\sum_{s=1}^K v_s c_s\right] = C_1 E\left[\sum_{s=1}^K e^{-g_1 s} c_s\right]$ and

$$E\left[\left(\sum_{s=1}^K v_s c_s\right)^2\right] = E\left[C_2 \sum_{s=1}^K e^{-g_2 s} c_s^2 + 2B \sum_{r=2}^K \sum_{s=1}^{r-1} e^{-g_1 s} e^{-(g_2 - g_1)r} c_s c_r\right].$$

3.4 Distribution of the Force of Interest

Under the simplifying assumption that $\varepsilon \sim N(0, \sigma^2)$, it is then straightforward to show that Δ_k follows a normal distribution with mean μ and variance $\sigma^2(1 + a_1^2 + \dots + a_q^2)$.

We then note that increasing q leads to an increase in the variance of Δ_k .

3.5 Proposition 3

- (a) If $\varepsilon \sim N(0, \sigma^2)$ and $0 < a_1 + a_2 + \dots + a_q < 1$, then
- i) $g_1 < d_1$
 - ii) $g_2 < d_2$

- (b) If $\varepsilon \sim N(0, \sigma^2)$ and $\sum_{j=0}^{q-1} S_j (S_q - S_j) > 0$ then $C_n < 1$ for $n \geq 1$ and $B < 1$ (using the notation of Proposition 2).

Proof

- (a) i) $d_1 = \mu - \sigma^2/2$. Then,

$$\begin{aligned} g_1 &= \mu - \ln(M(-(1 + a_1 + \dots + a_q))) = \mu - \ln(e^{\frac{1}{2}\sigma^2(1 + \dots + a_q)^2}) = \\ &= \mu - \frac{1}{2}\sigma^2(1 + \dots + a_q)^2. \end{aligned}$$

$$g_1 - d_1 = -\frac{1}{2}\sigma^2 [(1 + \dots + a_q)^2 - 1] .$$

Then, we note that $0 < a_1 + a_2 + \dots + a_q \Rightarrow g_1 < d_1$.

Remark:

If we take, for example, the cases of an MA(1) or MA(2) model we can also say that a positive autocorrelated environment leads to $g_1 < d_1$. To demonstrate this point we define ρ_i for $i=1,2,\dots$ to be the autocorrelation function of the process. Then for:

• MA(1)

$$\rho_1 = a_1/(1+a_1^2) \text{ and } \rho_1 > 0 \Rightarrow a_1 > 0$$

• MA(2)

$$\rho_1 = (a_1 + a_1 a_2)/(1 + a_1^2 + a_2^2) \text{ and } \rho_1 > 0 \Rightarrow a_1(1 + a_2) > 0 \quad (i)$$

$$\rho_2 = a_2/(1 + a_1^2 + a_2^2) \text{ and } \rho_2 > 0 \Rightarrow a_2 > 0 \quad (ii)$$

Then (i), (ii) $\Rightarrow a_1 > 0, a_2 > 0 \Rightarrow a_1 + a_2 > 0$.

This indicates that the actuary should use a higher interest assumption than for the corresponding i.i.d environment.

ii) $d_2 = 2\mu - 2\sigma^2$. Then,

$$g_2 = 2\mu - \ln(M(-2\lambda)) \text{ where } \lambda = 1 + \sum_{i=1}^q a_i > 1 \quad (\text{because } \sum_{i=1}^q a_i > 0)$$

$$g_2 = 2\mu - \ln(e^{\frac{1}{2}\sigma^2 4\lambda^2}) = 2\mu - 2\sigma^2 \lambda^2$$

$$g_2 - d_2 = 2\sigma^2 - 2\sigma^2 \lambda^2 = 2\sigma^2(1 - \lambda^2) < 0 \Rightarrow g_2 < d_2 .$$

(b) With $S_0 = 1$ and $S_j = 1 + \sum_{i=1}^j a_i$ for $j \geq 1$, it is straightforward to show that, if $\varepsilon \sim N(0, \sigma^2)$ so

$$M(t) = e^{\frac{1}{2}t^2\sigma^2}, \text{ then}$$

$$\log C_1 = -\sigma^2 \sum_{j=0}^{q-1} S_j (S_q - S_j)$$

$$\log C_n = n^2 \log C_1 \text{ for } n \geq 1$$

and $\log B = 3 \log C_1$.

Then, if by assumption $\sum_{j=0}^{q-1} S_j (S_q - S_j) > 0$, $C_n < 1$, for $n \geq 1$, and $B < 1$.

As remarked by Frees (1990, page 104) (but using the notation of this paper) “...the interpretation is that in the case of reserves we have off-setting factors. For example in a

positively autocorrelated interest environment $a_i > 0$ thus $C_i < I$ and $g_i < d_i$. Now in general a lower interest factor means that the reserve is higher. However this is slightly offset in the calculation of reserves because we multiply by C_i , a factor less than one."

It is important to note that, because $\Delta \sim N(\mu, \sigma^2(I+a_1^2+\dots+a_q^2))$, if $\varepsilon \sim N(0, \sigma^2)$ then we can use the results that we have obtained in section 2 and utilise the formulae derived there to calculate the mean, standard deviation, skewness and kurtosis for the random present values of standard insurance and annuity contracts.

3.6 Examples

Tables 3.1 - 3.3, on the following pages, present the calculated values for the moments in the cases of a temporary insurance, endowment and whole life assurances in the MA(1) case (with $a_1=0.5$) with $\mu=0.07$ for different choices of x, n (where appropriate) and σ .

It is clear from the examples that, if we compare an $MA(q)$ model with $\varepsilon \sim N(0, \sigma^2)$ and parameters $\{a_k\}$ as in equation (3.1) and an i.i.d model with $\Delta_k \sim N(\mu, \sigma^2)$, the standard deviations of the actuarial functions in the MA(q) model are higher than for the $\Delta_k \sim N(\mu, \sigma^2)$ model. This result follows because the MA(q) model is also an $N(\mu, \sigma_q^2)$ with variance σ_q^2 , given by $\sigma^2(I+a_1^2+\dots+a_q^2)$ which is greater than σ^2 . So the variance in $MA(q)$ is higher than for the equivalent $N(\mu, \sigma^2)$. Also, as we mentioned in the previous section, an increase in σ_q will lead to a decrease in the skewness and kurtosis as demonstrated in Tables 3.1 - 3.3.

MA(1) Model

Temporary Insurance

$\mu = 0.07$

$a_1 = 0.5$

Age

20	30	40	20	30	40
----	----	----	----	----	----

Term

$\sigma = 0$

$\sigma = 0.03$

25	0.008802	0.015809	0.025669	0.00885	0.015956	0.02601
30	0.013122	0.025255	0.040547	0.013203	0.025507	0.041099
35	0.022484	0.042475	0.064463	0.022628	0.042899	0.065316
40	0.039462	0.070528	0.099116	0.039713	0.071211	0.100355
45	0.067541	0.112467	0.144796	0.067959	0.113508	0.14646

Expectation

25	0.06884	0.076295	0.079364	0.06943	0.077261	0.080651
30	0.080006	0.090104	0.092185	0.080811	0.091432	0.0939
35	0.102151	0.112944	0.111433	0.103243	0.114686	0.113573
40	0.133218	0.141417	0.133424	0.134655	0.143602	0.135953
45	0.170499	0.171063	0.154938	0.172335	0.173698	0.157808

Standard Deviation

25	8.953197	6.755749	5.724027	8.986409	6.743711	5.667181
30	6.969304	4.922426	4.16433	7.005533	4.922492	4.127158
35	5.097343	3.55962	3.145786	5.130624	3.565309	3.119783
40	3.683315	2.603871	2.498685	3.710288	2.609239	2.475344
45	2.630119	1.902682	2.058647	2.651816	1.906336	2.035061

Skewness

25	91.00436	58.89308	47.46284	91.57622	58.51052	46.49361
30	56.76734	33.43919	27.64519	57.24011	33.25586	27.06788
35	31.20699	18.67758	17.00241	31.5379	18.61273	16.67545
40	17.07158	11.08223	11.54311	17.26828	11.04479	11.3272
45	9.522994	7.047866	8.39353	9.635416	7.014803	8.239292

Kurtosis

$\sigma = 0.05$

$\sigma = 0.07$

25	0.008937	0.016223	0.026631	0.00907	0.016633	0.027598
30	0.013349	0.025962	0.042103	0.013572	0.026664	0.043664
35	0.022887	0.043665	0.066868	0.023281	0.044844	0.069278
40	0.040163	0.072445	0.102605	0.040849	0.074346	0.106096
45	0.068712	0.115388	0.149478	0.06986	0.11828	0.154152

Expectation

25	0.070504	0.079043	0.083062	0.07218	0.081889	0.086996
30	0.082278	0.093885	0.097108	0.084568	0.097797	0.102332
35	0.105233	0.117903	0.117574	0.108339	0.123029	0.124084
40	0.137273	0.147636	0.140685	0.14136	0.154066	0.148385
45	0.175681	0.178563	0.163177	0.180905	0.186324	0.171914

Standard Deviation

25	9.054062	6.733352	5.577975	9.178035	6.748013	5.478238
30	7.078501	4.933583	4.073612	7.21002	4.979339	4.027539
35	5.196684	3.584425	3.084428	5.313443	3.636765	3.060715
40	3.78356	2.626172	2.44337	3.857086	2.670927	2.420991
45	2.694616	1.919302	2.001949	2.769617	1.955609	1.97583

Skewness

25	92.83429	58.02181	44.92987	95.39999	57.859	43.07325
30	58.28404	33.08919	26.18473	60.41524	33.29791	25.27376
35	32.25378	18.59865	16.19135	33.68066	18.86745	15.74431
40	17.69189	11.04091	11.00921	18.53188	11.21512	10.72259
45	9.879167	6.997242	8.012467	10.36652	7.089859	7.810302

Kurtosis

(Table 3.1)

MA(1) Model

Endowment Assurance

		$\mu = 0.07$			$a_1 = 0.5$					
Age	Term									
		20	30	40	20	30	40			
		$\sigma = 0$			$\sigma = 0.03$					
Expectation	25	0.250732	0.130961	0.075913	0.253517	0.133068	0.077398			
	30	0.252214	0.135393	0.08456	0.255001	0.137519	0.086114			
	35	0.255969	0.144352	0.099852	0.258755	0.14651	0.101511			
	40	0.263217	0.159945	0.124037	0.265999	0.16215	0.125843			
	45	0.275708	0.184779	0.159124	0.278481	0.18705	0.161114			
Standard Deviation	25	0.040116	0.054966	0.065193	0.055072	0.060021	0.067334			
	30	0.043825	0.062549	0.075287	0.057902	0.067289	0.077583			
	35	0.054743	0.07873	0.093565	0.066663	0.082914	0.095996			
	40	0.0721	0.101666	0.117287	0.081673	0.105423	0.119901			
	45	0.094606	0.128931	0.143257	0.102284	0.13244	0.148115			
Skewness	25	12.16509	9.478377	7.736863	4.879921	7.471243	7.237005			
	30	10.2824	7.407445	5.818319	4.636545	6.146473	5.523126			
	35	7.768355	5.463463	4.315624	4.449238	4.844685	4.164914			
	40	5.698423	4.026704	3.264449	4.027286	3.741725	3.183879			
	45	4.171295	2.977109	2.499181	3.375243	2.84175	2.450719			
Kurtosis	25	165.9879	106.9121	76.5245	49.25991	78.07852	69.89253			
	30	122.6754	68.83172	46.31082	42.80372	53.8772	43.09124			
	35	71.7015	38.79347	26.66794	34.77302	33.26081	25.39263			
	40	39.41627	21.9352	16.07935	25.49944	20.0582	15.54933			
	45	21.8678	12.83286	10.20655	17.03316	12.18197	9.958938			
		$\sigma = 0.05$			$\sigma = 0.07$					
Expectation	25	0.258547	0.1369	0.080116	0.266282	0.142865	0.08439			
	30	0.260031	0.141386	0.088955	0.267767	0.1474	0.093414			
	35	0.263783	0.150431	0.10454	0.271514	0.156524	0.109279			
	40	0.27102	0.166154	0.129133	0.278737	0.172367	0.134264			
	45	0.283487	0.19117	0.16473	0.291175	0.19755	0.170347			
Standard Deviation	25	0.076202	0.069126	0.071506	0.07218	0.081889	0.086996			
	30	0.078367	0.075941	0.082013	0.084568	0.097797	0.102332			
	35	0.0852	0.0907	0.100646	0.108339	0.123029	0.124084			
	40	0.097617	0.112452	0.124862	0.14136	0.154066	0.148385			
	45	0.11571	0.139049	0.151503	0.180905	0.186324	0.171914			
Skewness	25	2.282594	5.262726	6.426361	1.619904	3.642431	5.422903			
	30	2.294437	4.641331	5.044494	1.636954	3.429743	4.451795			
	35	2.488333	4.009545	3.919068	1.753803	3.223358	3.612668			
	40	2.624981	3.320969	3.053758	1.889946	2.872383	2.89575			
	45	2.516327	2.633357	2.374725	1.921062	2.398227	2.288712			
Kurtosis	25	16.2816	48.07694	59.23725	8.510926	27.2527	46.17849			
	30	15.58845	36.62766	37.90461	8.459048	23.15619	31.51925			
	35	15.58903	25.85158	23.32122	8.84362	18.88824	20.75796			
	40	14.55244	17.26216	14.69841	9.040327	14.23614	13.68631			
	45	11.90856	11.16576	9.574239	8.391063	10.00475	9.159518			

(Table 3.2)

Whole Life Assurance

MA(1) Model

$$\mu = 0.07$$

$$a_1 = 0.5$$

Age
▼

$\sigma = 0$		$\sigma = 0.03$	
25	0.046933	0.047885	Expectation
30	0.062972	0.06414	
35	0.085592	0.087004	
40	0.118088	0.117758	
45	0.155641	0.157574	
25	0.0739	0.075324	Standard Deviation
30	0.083739	0.08555	
35	0.100691	0.102759	
40	0.122126	0.124625	
45	0.145857	0.148878	
25	8.241921	8.104008	Skewness
30	4.808983	4.704835	
35	3.769096	3.696432	
40	3.008606	2.955872	
45	2.398801	2.359019	
25	55.43961	53.50037	Kurtosis
30	34.97892	33.74101	
35	22.08254	21.40668	
40	14.50455	14.12856	
45	9.765005	9.55051	
$\sigma = 0.05$		$\sigma = 0.07$	
25	0.049635	0.052408	Expectation
30	0.066282	0.069659	
35	0.089585	0.093634	
40	0.120806	0.125564	
45	0.161087	0.166545	
25	0.078065	0.082735	Standard Deviation
30	0.089012	0.094846	
35	0.106879	0.113756	
40	0.12935	0.137175	
45	0.153982	0.1627	
25	5.86464	5.530547	Skewness
30	4.531234	4.30673	
35	3.57861	3.434947	
40	2.872314	2.775742	
45	2.297662	2.231308	
25	50.15696	45.54467	Kurtosis
30	31.69289	29.08251	
35	20.3174	19.01231	
40	13.53592	12.86878	
45	9.221833	8.88314	

(Table 3.3)

4. CONCLUDING COMMENTS

As noted by Frees (1990) and Dufresne (1992), moving average processes often lead to tractable results and are simpler to manipulate than the full ARMA processes while still incorporating dependence over time. This arises because of the relatively simple form of the covariance structure. In this paper, we demonstrate the tractability and the convenience in the case of standard present value calculations in a life insurance context. There is a duality between the standard AR and MA models which means that, in practice, it is often difficult to distinguish between them when fitting models to observational data (Frees (1990)). Indeed, any lack of fit with actual data from using MA(q) models may be offset by the simplifications arising from their use.

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